3.2 Rolle's Theorem and the Mean Value Theorem

**Thm. 3.3 Rolle's Theorem**

→ gives conditions that guarantee the existence of an extreme value in the interior of a closed interval

Let \( f \) be continuous on \([a, b]\) and differentiable (smooth) on \((a, b)\).

If \( f(a) = f(b) \), then there is at least one number \( c \) between \( a \) and \( b \) where \( f'(c) = 0 \).

English translation:

"a smooth curve cannot intersect a horizontal line twice without having a horizontal tangent somewhere in between"

Note:

If the differentiability requirement is dropped from Rolle's Thm., \( f \) will still have a critical number in \((a, b)\), but it may not yield a horizontal tangent.
Find the zeros (x-intercepts) of \( f(x) = x^2 - 3x + 2 \) and show that \( f'(x) = 0 \) at some point between the two intercepts. (find \( c \))
Theorem 3.4: The Mean Value Theorem

Geometric interpretation:

If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is at least one number \( c \) between \( a \) and \( b \) such that

\[
m_{sec} = \frac{f(b) - f(a)}{b - a} = m_{tan} @ x = c = f'(c)
\]

Definition:

If \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there is at least one number \( c \) between \( a \) and \( b \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

English translation:

"Between any two points of a smooth curve, there is a point at which the tangent is \( \parallel \) to the chord joining the points!"

Note:
The "mean" in the M.V.T. refers to the mean (or average) rate of change of \( f \) in the interval \([a, b]\).

The M.V.T. is used more often to prove other theorems!

The M.V.T. is considered by many to be [the most important theorem] in Calculus!!
The M.V.T. implies that there must be a point in \((a, b)\) at which the instantaneous rate of change is equal to the average rate of change over the interval \([a, b]\).

see Example 4 p171 [speeding ticket!]
If Rolle's Thm. can be applied, find all value(s) of \( c \) in the open interval \((a, b)\) such that \( f'(c) = 0\).

\[ f(x) = \cos x, \quad [0, 2\pi] \]

\[ f(0) = \cos 0 = 1 \]

\[ f(2\pi) = \cos 2\pi = 1 \]

\[ f \text{ is differentiable on } (0, 2\pi); \text{ everywhere!} \]

\[ f'(x) = -\sin x \]

Set \( f'(x) = 0 \),

\[- \sin x = 0\]

\[ \sin x = 0 \]

\[ x = \sin^{-1}(0) \text{ on } (0, 2\pi) \]

\[ x = 0, \pi, 2\pi, \ldots \]

\[ \therefore c = \pi \]
If the Mean Value Theorem can be applied, find all value(s) of \( c \) in the open interval \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}
\]

\[
f(x) = \frac{x + 1}{x}, \quad [\frac{1}{2}, 2]
\]

\[
f\left(\frac{1}{2}\right) = \frac{\frac{1}{2} + 1}{\frac{1}{2}} = \frac{3/2}{1/2} = \frac{3}{2} \cdot \frac{2}{1} = 3 = f(a)
\]

\[
f(2) = \frac{2 + 1}{2} = \frac{3}{2} = f(b)
\]

\[
f'(x) = \frac{(x(1)) - (x+1)(1)}{(x)^2} = \frac{x - x - 1}{x^2} = \frac{-1}{x^2}, \quad x \neq 0
\]

\[
\Rightarrow f'(c) = -\frac{1}{c^2}
\]

\[
-\frac{1}{c^2} = \frac{\frac{3}{2} - 3}{2 - \frac{1}{2}}
\]

\[
-\frac{1}{c^2} = \frac{-3/2}{3/2}
\]

\[
-\frac{1}{c^2} = -1
\]

\[
c^2 = 1
\]

\[
c = \pm \sqrt{1} = \pm 1
\]

\[
\text{on (1/2, 2) interval}
\]

\[
C = 1
\]
Assignment:

p172-173 #1, 2, 3-15 odd, 25, 29, 30, 31-37 odd, 43, 45, 47-49